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PARSIMONIOUS STAGGERED GRID FINITE-DIFFERENCING OF THE WAVE EQUATION

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Abstract. A parsimonious staggered grid differencing scheme is presented which requires less storage than the conventional staggered grid method. For three dimensional elastic wave propagation, this scheme only stores displacement components, not stress, and so requires 66% of the memory needed by the standard staggered grid method. The storage requirement is the same as the 2-2 differencing scheme used by Kelly et al. (1976) for the second-order wave equation. Its advantage is that it is stable and accurate for media with fluid-elastic contacts and for a wide range of Poisson ratios. A disadvantage is that its computer programming is more involved.

Introduction

A major problem with some finite-difference solutions of the wave equation is that they become unstable or inaccurate for a range of Poisson's ratios, typically larger than .25 (Levander, 1988; Marfurt, 1984) Thus, these finite-difference schemes are not applicable to problems involving weathering zones with large Poisson ratios or across fluid-solid contacts such as the sea bottom. An example is the 2-2 scheme of Kelly et al. (1976) which will be denoted as the standard 2-2 finite-difference scheme. To overcome this problem, staggered grid finite-difference schemes have been proposed for the coupled first-order equations of stress-strain and conservation of momentum (Madariaga, 1976; Virieux, 1984; and Levander, 1988). Other workers, notably Yee (1966) used this method for the equivalent first-order equations in electromagnetic theory, namely Maxwell's equations. The staggered grid method can accommodate large gradients in physical parameters because, in part, 1). central differencing of 1st-order field derivatives are naturally centered around the grid point of the physical parameter, and 2). there are no explicit spatial derivatives of physical parameters; coupled first-order stress-strain and momentum equations do not contain derivatives of physical parameters.

The major drawback with staggered grid methods is that they require more memory than the conventional differencing schemes (e.g., Kelly et al., 1976). For a three-dimensional problem, the conventional 2-2 differencing of the second-order elastic wave equation computes only $3N^3$ displacement values at each time step, where N is the number of grid points along an edge of a gridded cube model. This means that two cubes of grid points ($6N^3$) need to be stored at each time step. In contrast, the 2-2 staggered grid method computes 6 stress components and 3 velocity components every two half time steps. This means that a cube of stress components (6 stresses per grid point) and a cube of velocity components (3 components per grid point) need to be stored at any one time step, for a storage demand of $9N^3$ field values per time step. Therefore, a 2-2 staggered grid scheme for three-dimensional elastic wave propagation requires 1.5 times more storage than that using the conventional 2-2 differencing of the second-order wave equation. This storage demand can severely limit the model size for staggered grid methods.

Parsimonious Staggered Grid Method

To overcome the storage limitations of the staggered grid method, we propose a parsimonious staggered grid method. Parsimonious staggered grid differencing only requires, for a three-dimensional problem, the storage of two cubes of displacement components per time step, where there are three displacement components at each grid point. Hence, the storage requirements for the parsimonious staggered grid scheme are the same as for the standard 2-2 finite-differencing scheme. It also enjoys the advantage of being stable for fluid-solid interfaces and for a wide range of Poisson ratios. It can be shown by induction that the parsimonious staggered grid method is mathematically equivalent to the staggered grid method.

Theory

For two-dimensions, the conservation of momentum equation are

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{1a}$$

$$\rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

and the stress-strain relations are

$$\tau_{xx} = (\lambda+2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} \tag{1b}$$

$$\tau_{yy} = (\lambda+2\mu)\frac{\partial v}{\partial y} + \lambda\frac{\partial u}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})$$

where λ and μ are Lame's parameters, ρ is density, and u and v are displacement components in the x and y directions respectively. The staggered grid scheme reparameterizes $\dot{u}=\frac{\partial u}{\partial t}$ and differentiates equation (1b) with respect to time, so that equations (1) become

$$\rho\frac{\partial \dot{u}}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{2a}$$

$$\rho\frac{\partial \dot{v}}{\partial t} = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda+2\mu)\frac{\partial \dot{u}}{\partial x} + \lambda\frac{\partial \dot{v}}{\partial y} \tag{2b}$$

$$\frac{\partial \tau_{yy}}{\partial t} = (\lambda+2\mu)\frac{\partial \dot{v}}{\partial y} + \lambda\frac{\partial \dot{u}}{\partial x}$$

$$\frac{\partial \tau_{xy}}{\partial t} = \mu(\frac{\partial \dot{v}}{\partial x} + \frac{\partial \dot{u}}{\partial y})$$

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The standard 2-2 staggered grid scheme used by, say Virieux (1986), assigns the velocity and stress components to the grid points shown in Figure (1). The 1st-order spatial derivatives of stress and velocity in equation (2) are approximated by 2nd order-correct centered difference approximations, e.g.,

$$\frac{\partial \dot{u}(i+1/2\Delta x, j\Delta y, t)}{\partial x} \approx \frac{\dot{u}^t{}_{i+1j} - \dot{u}^t{}_{ij}}{\Delta x}$$

$$\frac{\partial \tau_{xx}(i\Delta x, j\Delta y, t)}{\partial x} \approx \frac{(\tau_{xx})_{i+1/2j}{}^t - (\tau_{xx})_{i-1/2j}{}^t}{\Delta x}$$

where Δx and Δy are the distances between grid points in the x and y direction respectively. The i and j indices correspond to the grid point numbering in the x and y directions respectively.

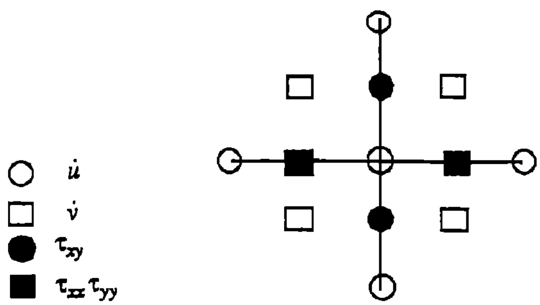

Fig. 1. The grid points at which the stress and particle velocity components are spatially staggered. The $\dot{u}$ components are restricted to the (i $\Delta x, j\Delta y$) grid points, the $\dot{v}$ components are restricted to the (i+1/2 $\Delta x, j$+1/2 Δy) grid points, the $\dot{\tau}_{xy}$ components are restricted to the (i $\Delta x, j$+1/2 Δy) grid points, and the $\dot{\tau}_{yy}$ and $\dot{\tau}_{yy}$ are restricted to the (i+1/2 $\Delta x, j\Delta y$) grid points. The parsimonious staggered grid scheme is the same as above except that the displacement components take the place of velocity components.

The time derivatives in equation (2) are replaced by centered differences in time, except the stress time derivatives are centered at time t and the velocity time derivatives are centered at time t+1/2, e.g.,

$$\frac{\partial \dot{u}(i\Delta x, j\Delta y, t+1/2)}{\partial t} \approx \frac{\dot{u}_{ij}{}^{t+1} - \dot{u}_{ij}{}^{t}}{\Delta t}$$

$$\frac{\partial \tau_{xx}(i\Delta x, j\Delta y, t)}{\partial t} \approx \frac{(\tau_{xx})_{ij}{}^{t+1/2} - (\tau_{xx})_{ij}{}^{t-1/2}}{\Delta t}$$

The differencing scheme for the x displacement component in equation (2a) is shown in Figure (2) where the stress components are confined to the t+(2n+1)/2 time panels and the velocity components are confined to the t+n time panels, where n can take on integer values. Stress and velocity panels are staggered in this way for all time steps.

Thus, for any one time step one panel of stress components and one panel of velocity components need to be stored. For three dimensional problems, this means that a staggered grid scheme requires 1.5 times more storage than the conventional 2-2 differencing scheme for the second-order elastic wave equation.

To overcome this storage limitation, we propose a parsimonious staggered grid formulation. The parsimonious staggered grid scheme solves for displacement components using equation

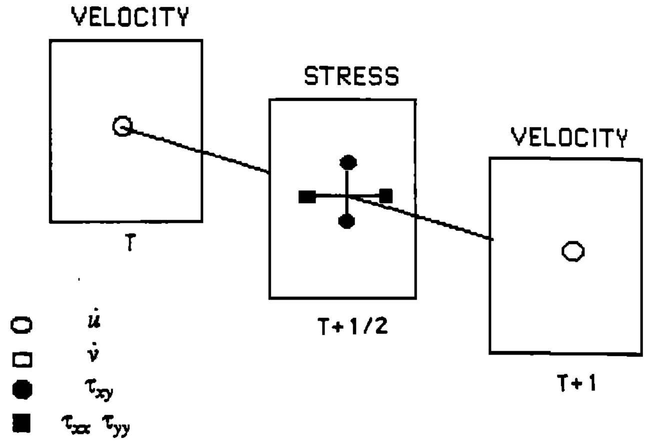

Fig. 2. Stencil for time stepping the $\dot{u}$ velocity component by the standard staggered grid method. The first-order equations in equation (2) are stepped forward in time by using the stresses at time T+1/2 and velocities at time T to compute the velocities at time T+1. The stresses at time T+1.5 are computed using the velocities at time T+1 and stresses at time T+1/2. Note that at any one time step a panel of stresses and velocities need to be stored.

(1). The following is a 2-2 parsimonious staggered grid algorithm;

1). Replace the time derivatives in equation (1a) by second-order correct centered differencing, shown in Figure (3). Assume the pivot is at time t and the unknown displacements are at the t+1 panel.

2). The stress values at the pivot time t on the right hand side of equation (1a) are computed by using the displacement components (at the time t panel) in the stress-strain equations in equation (1b). The spatial derivatives on the RHS of equation (1b) are approximated by central difference formulae. The grid point locations for the stresses and displacements are the same as the standard staggered grid method in Figure (1).

3). The stress values (at the time t panel) computed in step 2 are then used to compute stress gradients (at time t panel) by centered differencing formulae.

4). Using the stress gradients at time t in step 3 and the displacements at time t-1, the displacements are computed at the t+1 panel. Steps 1-4 are repeated after incrementing the time step.

To minimize storage, steps 2 and 3 are combined so that only one differencing formula is used to compute

$$u_{ij}{}^{t+1} = 2u_{ij}{}^{t} - u_{ij}{}^{t-1} + \frac{\Delta t^2}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) \quad (3a)$$

and

$$v_{ij}{}^{t+1} = 2v_{ij}{}^{t} - v_{ij}{}^{t-1} + \frac{\Delta t^2}{\rho}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) \quad (3b)$$

where the stresses are computed from centered differences of equation (1b), e.g.,

$$\tau_{xx}(i+1/2\Delta x, j\Delta y, t) = (\lambda+2\mu)\frac{u^t_{i+1j} - u^t_{ij}}{\Delta x} + \lambda\frac{v^t_{i+1/2j+1/2} - v^t_{i+1/2j-1/2}}{\Delta y} \quad (4a)$$

$$\tau_{xy}(i\Delta x, j+1/2\Delta y, t) = \mu\left(\frac{u^t_{ij+1} - u^t_{ij}}{\Delta y} + \frac{v^t_{i+1/2j+1/2} - v^t_{i-1/2j+1/2}}{\Delta x}\right) \quad (4b)$$

The spatial derivative operators in equation (3a) or (3b) are also computed by centered differences, e.g.,

$$\frac{\partial\tau_{xx}(i\Delta x, j\Delta y, t)}{\partial x} = \frac{(\tau^t_{xx})_{i+1/2j} - (\tau^t_{xx})_{i-1/2j}}{\Delta x} \quad (5a)$$

$$\frac{\partial\tau_{xy}(i\Delta x, j\Delta y, t)}{\partial y} = \frac{(\tau^t_{xy})_{ij+1/2} - (\tau^t_{xy})_{ij-1/2}}{\Delta y} \quad (5b)$$

Equations (4) are substituted into (5), and these in turn are substituted into (3) to yield the differencing formula in terms of displacement only (see Figure 3). This combination of steps 2 and 3 is the reason for parsimony, stresses do not have to be stored. Only two panels of displacement need to be stored at any one time. In addition it retains the beneficial property of centering all space derivatives in the same manner as the standard staggered grid formulation. In fact, the two schemes are equivalent.

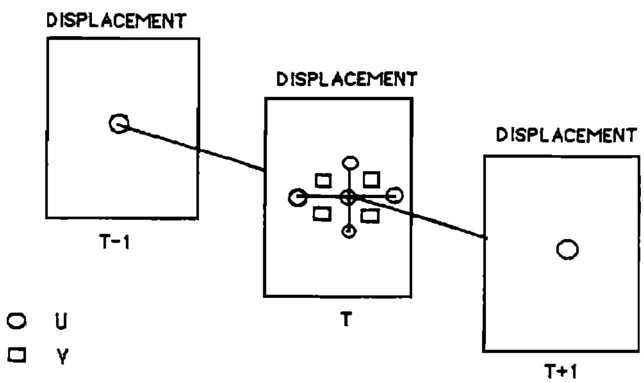

Fig. 3. Parsimonious staggered grid method. Equations (1) are stepped forward in time by using the displacements at time T and T-1 to compute the displacements at time T+1. The stress gradients at time T on the right hand side of equation (1a) are computed using the differenced stress strain equations in equation (1b). The stencil in this figure is for the *u* displacement component.

Numerical Example

The line source response of a water layer over an elastic salt dome model Figure (4a) is computed by the standard staggered grid method and compared to the parsimonious staggered grid solution in Figures (4b) and (4c). The two solutions are identical to five significant figures (Figure 4c). The standard 2-2 scheme of Kelly et al. (1976) would fail in this circumstance.

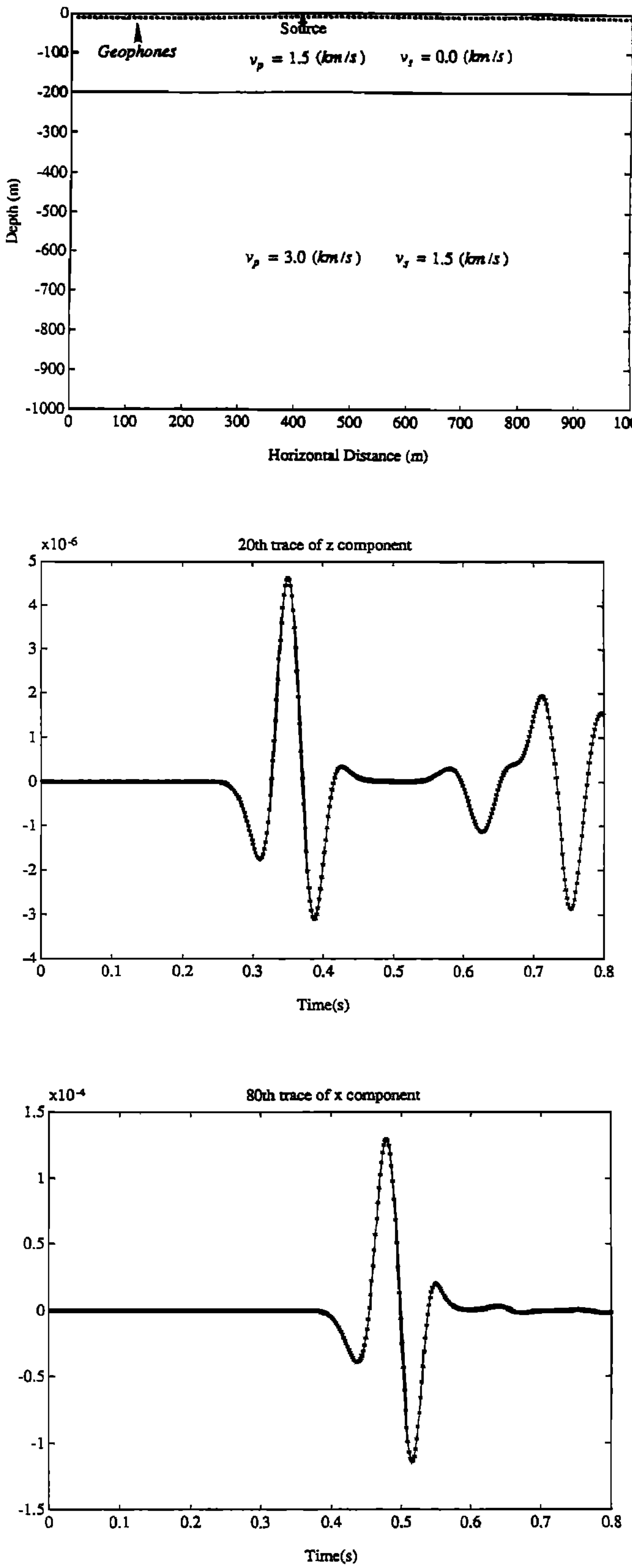

Fig. 4. (a) Layered salt dome model underlying a water layer with a line source in the water layer. Comparison between the standard (*) and parsimonious (-) seismograms at trace 20 (b) and trace 80 (c). The geophone numbering is from left to right in Figure 4a.

Conclusions

A 2-2 parsimonious staggered grid scheme is presented which only requires storage of two displacement panels at any one time step. This is the same storage requirement as the standard 2-2 differencing of the second-order wave equation, and, for a three-dimensional elastic problem, 66% that of a standard staggered grid method. Similar to the staggered grid method, it is accurate and stable for fluid-elastic interfaces and high Poisson ratios. With clever programming, the computational cost of the parsimonious scheme should be approximately the same as the standard staggered grid scheme. The advantage is that the parsimonious scheme can accommodate much larger models than the standard staggered grid scheme. Its disadvantage is that its computer programming is not as easy. Extensions to higher order differencing schemes are straightforward.

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