Skeletonized Wave-Equation Dispersion
Inversion: Dispersion Misfit Function

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August 23, 2015

Abstract
I present the theory for skeletonized wave equation inversion of dispersion curves using a dispersion misfit function. Similar to wave equation traveltime inversion, the complicated surface-wave arrivals in traces are skeletonized as simpler data, namely the picked dispersion curves in the \((k_x, \omega)\) domain. The elastic wave equation is then used to invert these curves for 2D or 3D velocity models where the objective function is the sum of the squared residuals; here the residual is the difference between the wavenumbers along the predicted and recorded dispersion curves. This procedure, denoted as skeletonized wave equation dispersion inversion (WD), does not require the assumption of a layered model and is less prone to the cycle skipping problems of full waveform inversion (FWI).

Introduction
Zhang and Schuster (2014) presented an iterative gradient method to invert dispersion curves for 2D or 3D S-velocity models. This procedure is more robust than FWI because it replaces complicated surface-wave arrivals with simple dispersion curves in the wavenumber \(k_x - \omega\) or phase-velocity \(C(\omega) - \omega\) domains in Figure 1. The merits of WD are that it achieves robust convergence for the models tested, is a simple method to code because it relies on finite-difference approximations to the Fréchet derivative, and is computationally inexpensive if there are not too many unknowns in the model. Its main liability is that it becomes prohibitively expensive if there are too many unknowns in the model, requiring a finite-difference solution of the elastic wave equation for every unknown model parameter and shot gather.
Figure 1: Common shot gather $d(x, t)$ on the left and the fundamental (n=0) dispersion curves for Rayleigh waves in the (top right) $k - \omega$ and (bottom right) $C(\omega) - \omega$ domains. Here the phase velocity is $C(\omega) = \omega/k(\omega)$ and $\kappa(\omega)$ is the skeletonized data.

To reduce this expense the WD algorithm can be reformulated so its objective function is the sum of squared dispersion residuals. This might enjoy better convergence properties than the WD objective function defined by the sum of the predicted energy spectrum along the observed dispersion curve (Schuster, 2015).

1. **Misfit function.** The dispersion misfit function $\epsilon$ is defined as the sum of squared dispersion residuals:

$$\epsilon = \frac{1}{2} \sum_{\omega} (\kappa(\omega) - \kappa(\omega)^{\text{obs}})^2,$$

so that the optimal shear-slowness model $s(x)$ is obtained from the iterative steepest descent solution:

$$s(x)^{(k+1)} = s(x)^{(k)} - \alpha \gamma(x),$$

$$= s(x)^{(k)} - \alpha \sum_{\omega} \Delta \kappa(\omega) \frac{\partial \kappa(\omega)}{\partial s(x)},$$

where $\alpha$ is the step length and the superscript $(k)$ denotes the $k^{th}$ iteration. For pedagogical simplicity we have assumed a single shot gather, but the misfit function can include a summation over different shot gathers if more than one shot gather is used.
2. **Connective function.** The Fréchet derivative $\frac{\partial \kappa(\omega)}{\partial s(x)}$ in equation 2 is obtained by forming a connective function that relates the residual $\Delta \kappa(\omega)$ to the slowness model. This connective function is defined to be crosscorrelation between the predicted $\hat{\bar{D}}(k, \omega)_{obs}$ and observed $\hat{\bar{D}}(k, \omega)_{obs}^*$ spectra in the $(k, \omega)$ domain:

$$\dot{\Phi}(\Delta \kappa, s(x)) = \text{Real}\{ \int \hat{\bar{D}}(k + \Delta \kappa, \omega)_{obs}^* \hat{\bar{D}}(k, \omega) dk \} = 0,$$

where $\dot{\hat{\bar{D}}}(k, \omega)_{obs} = \frac{\partial \hat{\bar{D}}(k, \omega)}{\partial k}$ and $\Delta \kappa$ is the wavenumber lag that aligns the predicted and observed real part of the spectra with one another for a specified $\omega$. This wavenumber lag is also equal to the dispersion residual in equation 1, and so equation 3 connects the slowness model with the dispersion data.

3. **Fréchet derivative.** The implicit function theorem implies from equation 3 that $\kappa$ is an implicit function of $s(x)$ so that

$$d\dot{\Phi} = \frac{\partial \dot{\Phi}}{\partial s} ds + \frac{\partial \dot{\Phi}}{\partial \kappa} d\kappa = 0.$$  

(4)

Rearranging this equation gives the Fréchet derivative

$$\frac{\partial \kappa}{\partial s(x)} = -\frac{\partial \dot{\Phi}/\partial s}{\partial \dot{\Phi}/\partial \kappa},$$

(5)

where the denominator is the normalization term

$$A = \frac{\partial \dot{\Phi}(\Delta \kappa, s(x))}{\partial \kappa} = \text{Real}\{ \int \hat{\bar{D}}(k + \Delta \kappa, \omega)_{obs}^* \hat{\bar{D}}(k, \omega) dk \},$$

(6)

and the numerator is

$$\frac{\partial \dot{\Phi}(\Delta \kappa, s(x))}{\partial s(x)} = \text{Real}\{ \int \hat{\bar{D}}(k + \Delta \kappa, \omega)_{obs}^* \frac{\partial \hat{\bar{D}}(k, \omega)}{\partial s(x)} dk \}.$$  

(7)

Plugging equations 6-7 into equation 5 gives the Fréchet derivative:

$$\frac{\partial \kappa}{\partial s(x)} = -\frac{\partial \dot{\Phi}/\partial s}{\partial \dot{\Phi}/\partial \kappa} = -\frac{1}{A} \text{Real}\{ \int \hat{\bar{D}}(k + \Delta \kappa, \omega)_{obs}^* \frac{\partial \hat{\bar{D}}(k, \omega)}{\partial s(x)} dk \},$$

$$= -\frac{1}{A} \text{Real}\{ \int \hat{\bar{D}}(y, \omega)_{obs}^* \frac{\partial D(y, \omega)}{\partial s(x)} dy \}. $$

(8)

The last equation follows by equation 13 in the Appendix where $D(y, \omega)$ is the inverse Fourier transform of $\hat{\bar{D}}(k, \omega)$ and $\hat{\bar{D}}(y, \omega)_{obs}^*$ is defined in equation 14.
We recall the Born approximation

\[
\frac{\partial D(y, \omega)}{\partial s(x)} = -2s(x)W(\omega)G(y, \omega|x, 0)G(x, \omega|s, 0),
\]

(9)

where \(W(\omega)\) is the source-wavelet spectrum and \(G(y, \omega|x, 0)\) is the harmonic solution to the elastic wave equation for a vertical force at the point \(x\) and a vertical-component particle-velocity recording at \(y\); this harmonic solution only contains the fundamental mode for the Rayleigh wave. Recognizing that \(y = (y, 0)\) is associated with the geophone locations \(g = (x_g, 0)\) on the surface so that \(y \rightarrow g\), and plugging equation 9 into equation 8 gives the final form for the Fréchet derivative:

\[
\frac{\partial \kappa}{\partial s(x)} = 2s(x)\Delta \kappa(\omega) \frac{W(\omega)}{A} \int \left[ \hat{D}(g, \omega)_{obs} G(g, \omega|x, 0)^\ast \right] dx_g. \tag{10}
\]

4. **Gradient Update.** Plugging equation 10 into equation 2 gives the iterative formula for updating the S-velocity model from a single shot gather:

\[
s(x)^{(k+1)} = s(x)^{(k)} - \alpha \sum_\omega \frac{2s(x)\Delta \kappa(\omega)}{A} f(x, s)_{\omega} g(x, s)^\ast. \tag{11}
\]

It says that backprojected data \(g(x, s)_{\omega}\) are conjugated and multiplied by the weighted source field \(\frac{2s(x)\Delta \kappa(\omega)}{A} f(x, s)_{\omega}\) to give the slowness update at \(x\). In the space-time domain, the modified source wavelet is defined as \(\Delta \kappa(\omega)W(\omega)\) so that the slowness update is proportional to the zero-lag correlation between the backprojected data and the weighted source field. The \(\text{Real}\) label is dropped because a symmetrical summation over frequencies will automatically eliminate the imaginary part of the conjugate symmetric summand.

Thus, the slowness model is updated by calculating the magnitude spectra of the dispersion curves for both the recorded and computed data, then computing \(\Delta \kappa(\omega)\) for each shot gather. The recorded data \(D(g, \omega)_{obs}\) for each source are migrated, where the forward propagated source has the weighted source spectrum \(W(\omega)\Delta \kappa(\omega)\). The gradients for each migrated shot gather are added together to get the slowness update.

**Summary**

The theory for skeletonized wave equation inversion is developed for a dispersion misfit function. This theory says that the S-velocity model is updated by migrating
the weighted data, where the weight is proportional to the dispersion residual. It largely overcomes the expense of finding the Fréchet derivative by a finite-difference approximation. Numerical simulations will be carried out to test the effectiveness of this new procedure.

References


Appendix: Correlation Identity

In equation 7, the functions in the integrand can be replaced by their Fourier transforms:

\[
\hat{D}(k + \Delta \kappa, \omega)_{\text{obs}} = \frac{1}{2\pi} \int [iy' D(y', \omega)_{\text{obs}} e^{i(k + \Delta \kappa)y'}] dy',
\]

\[
\hat{D}(k, \omega) = \frac{1}{2\pi} \int D(y, \omega) e^{iy} dy,
\]

(12)

to give

\[
\frac{\partial \Phi(\Delta \kappa, s(x))}{\partial s(x)} = \frac{1}{4\pi^2} \text{Real} \left\{ \int dy \left\{ \int dy' \left[ \int e^{i(k(y - y') - \Delta \kappa y')} (-iy') D(y', \omega)_{\text{obs}}^* e^{-i\Delta \kappa y'} \right] D(y, \omega) \right\},
\]

\[
= \frac{1}{4\pi^2} \text{Real} \left\{ \int dy \left\{ \int dy' \left[ 2\pi \delta(y - y') \right] (-iy') D(y', \omega)_{\text{obs}}^* e^{-i\Delta \kappa y'} \right] D(y, \omega) \right\},
\]

\[
= \frac{1}{2\pi} \text{Real} \left\{ \int dy \left[ -yi \hat{D}(y, \omega)_{\text{obs}}^* e^{-i\Delta \kappa y} \right] D(y, \omega) \right\},
\]

\[
= \text{Real} \left\{ \int dy \left[ \frac{-iy e^{-i\Delta \kappa y}}{2\pi} D(y, \omega)_{\text{obs}}^* \right] D(y, \omega) \right\},
\]

(13)

where the weighted conjugated data function is

\[
\hat{D}(y, \omega)_{\text{obs}}^* = \frac{-iy e^{-i\Delta \kappa y}}{2\pi} D(y, \omega)_{\text{obs}}^*.
\]

(14)