Wave Equation Traveltime Inversion

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ABSTRACT

Connection between wave equation traveltime inversion and full waveform inversion is introduced. Numerical example of a square anomaly imbedded in a homogeneous medium is presented.

INTRODUCTION

Traditional traveltime tomography techniques assume infinite frequency bandwidth, and require traveltime picking for every trace. An eikonal solver is used in those methods, and its high-frequency approximation suffers from complex velocity structure. Meanwhile, full waveform inversion uses a wave equation solver, which handles velocity complexity naturally, but it is likely to stuck in local minima. Wave equation travel time inversion method uses a wave equation solver to get the Frechét derivative, and uses that to update the velocity model.

THEORY

First let's take a look at the theory behind full waveform inversion. In full waveform inversion, the misfit function is the squared l^2 norm of predicted (calculated) data and observed data:

$$\varepsilon = \frac{1}{2}\sum_{x_s,x_g}\int [p_{cal}(x_g,t;x_s) - p_{obs}(x_g,t;x_s)]^2 \mathrm{d}t. \quad (1)$$

We can change our slowness model s to minimize the misfit function in Equation 1. It is a non-linear optimization problem. Input s has $nz \times nx$ unknowns, where nz and nx are the number of points in vertical direction and horizontal direction respectively. Output $p_{cal}(x_g,t;x_s)$ has $nt \times ng \times ns$ variables. nt is the number of time samples. ng and ns are the number of geophones and shots respectively.

We can use any gradient based optimization method to solve this problem, for example, steepest descent method and conjugate gradient method. The key to this problem is to find the Frechét derivative, which describes how the misfit function ε changes when there is a perturbation in model s.

We denote the Frechét derivative as:

$$\frac{\delta\varepsilon}{\delta s}. \quad (2)$$

Note that only $p_{cal}(x_g,t;x_s)$ changes when we change the model s. The Frechét derivative mentioned above becomes:

$$\begin{aligned}\frac{\delta\varepsilon}{\delta s} &= \sum_{x_s,x_g}\int \frac{\delta p_{cal}(x_g,t;x_s)}{\delta s} \times \\ &\quad \times [p_{cal}(x_g,t;x_s) - p_{obs}(x_g,t;x_s)]\mathrm{d}t. \quad (3)\end{aligned}$$

After briefly introducing the theory of full waveform inversion, we now turn to the wave equation traveltime inversion. We generally follow the procedure introduced in Luo and Schuster (1991). It is also an optimization problem, the objective function related is as below:

$$\varepsilon = \frac{1}{2}\sum_{x_s,x_g}\tau_{max}(x_s,x_g)^2. \quad (4)$$

Where $\tau_{max}(x_s,x_g)$ is the time shift that maximizes the cross-correlation between calculated data $p_{cal}(x_s,t;x_g)$ and observed data $p_{obs}(x_g,t;x_s)$. The cross-correlation is defined as:

$$f(\tau,s) = \int p_{obs}(x_g,t+\tau;x_s)p_{cal}(x_g,t;x_s)\mathrm{d}t. \quad (5)$$

Where s is the slowness model.

Since at the maximum point $\tau_{max}(x_s,x_g)$, the derivative of $f(\tau,s)$ with respect to τ is always zero, it can be the bridge of $\tau_{max}(x_s,x_g)$ and slowness model s. This

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derivative has the following expression:

$$\frac{\mathrm{d}f(\tau,s)}{\mathrm{d}\tau} = \dot{f}(\tau,s) = \int \frac{\mathrm{d}p_{obs}(x_g,t+\tau;x_s)}{\mathrm{d}\tau} p_{cal}(x_g,t;x_s)\mathrm{d}t = \int \dot{p}_{obs}(x_g,t+\tau;x_s)p_{cal}(x_g,t;x_s)\mathrm{d}t. \quad (6)$$

Here comes the crucial step. We want to find how $\tau_{max}(x_s,x_g)$ changes when the slowness model s changes. Note that, at the maximum point, if we change the slowness model, $\dot{f}(\tau,s)$ is going to change. If we change τ, it is also going to change. If we want $\dot{f}(\tau,s)$ to remain zero, changes induced by these two variables need to cancel the effect of each other. Inspired by this idea, we get the implicit differentiation formula below:

$$\frac{\partial \tau_{max}}{\partial s} = -\frac{\frac{\partial \dot{f}(\tau,s)}{\partial s}|_{\tau=\tau_{max}}}{\frac{\partial \dot{f}(\tau,s)}{\partial \tau}|_{\tau=\tau_{max}}}. \quad (7)$$

The equation above applies to all shot-geophone pairs. Using the expression from Equation 6, we get the enumerator

$$\frac{\partial \dot{f}(\tau,s)}{\partial s}|_{\tau=\tau_{max}} = \int \frac{\partial p_{cal}(x_g,t;x_s)}{\partial s}\dot{p}_{obs}(x_g,t+\tau;x_s)\mathrm{d}t. \quad (8)$$

Similarly, we get the denominator, and we define its value as E

$$E = \frac{\partial \dot{f}(\tau,s)}{\partial \tau}|_{\tau=\tau_{max}} = \int \ddot{p}_{obs}(x_g,t+\tau;x_s)p_{cal}(x_g,t;x_s)\mathrm{d}t. \quad (9)$$

Keep in mind that in both expression, the value of τ is taken at the maximum point. Let's now rewrite Equation 7 as below:

$$\frac{\partial \tau_{max}}{\partial s} = \int \frac{\partial p_{cal}(x_g,t;x_s)}{\partial s}\frac{\dot{p}_{obs}(x_g,t+\tau;x_s)}{-E}\mathrm{d}t. \quad (10)$$

The last step is to find the Frechét derivative of objective function in Equation 4:

$$\frac{\partial \varepsilon}{\partial s} = \sum_{x_s,x_g}\frac{\partial \tau_{max}}{\partial s}\tau_{max} = \int \frac{\partial p_{cal}(x_g,t;x_s)}{\partial s} \times \times \frac{\tau_{max}\dot{p}_{obs}(x_g,t+\tau;x_s)}{-E}\mathrm{d}t. \quad (11)$$

If we compare Equation 11 with Equation 3, we can see the difference between full waveform inversion and wave equation traveltime inversion. In full waveform inversion, back-propagated signals are data residual, which are $p_{cal}(x_g,t;x_s) - p_{obs}(x_g,t;x_s)$. In wave equation traveltime inversion, back-propagated signals are observed traces after several process. The observed traces are first shifted by τ_{max} to best match predicted data, and time-derivatives are taken. Then they are weighted by corresponding traveltime errors τ_{max} and the minus reciprocal of second order time derivatives of cross-correlation function at τ_{max}

SYNTHETIC TEST

In the previous section, we discuss the theory of wave equation traveltime inversion, and compare it to full waveform inversion. For conciseness, realization of full waveform inversion Tarantola (1984) is not discussed here.

The model velocity is shown below, there is an anomaly in the center:

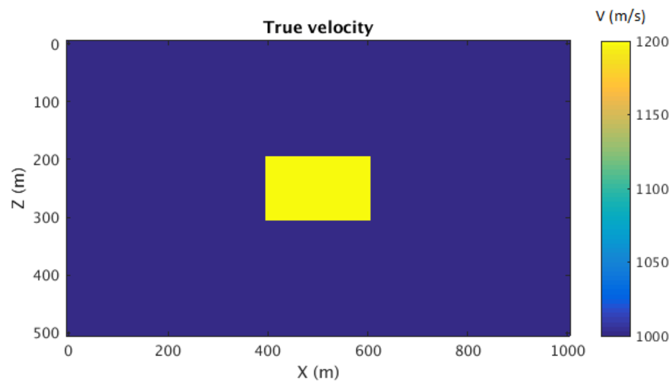

Figure 1: True velocity model

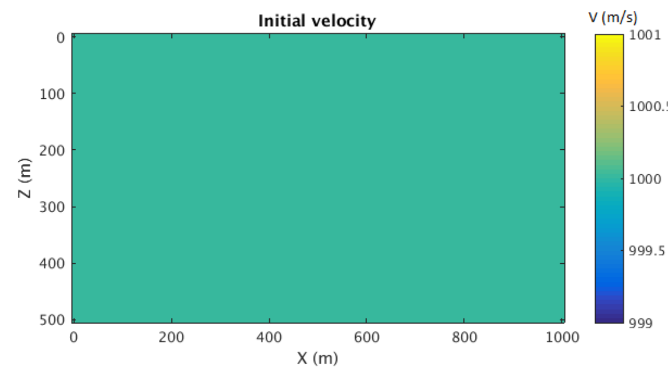

Figure 2: Initial velocity model

The model size is 150×400. There are 220 shots and 220 geophones. All shots/geophones are located around the model to gain maximum angle coverage. In the final result (Figure 3) We can see obvious artifacts near boundaries of the model. This is caused by strong amplitude in gradient near the shot location.

DISCUSSION

Wave equation traveltime inversion is essentially traveltime tomography. They are different in two aspects. First, traveltime tomography uses the first ground movement as a sign to determine the travel time, which is dominated by the high frequency part of the wave field, as pointed out by Maarten and van Der Hilst (2005). In wave equation traveltime inversion, a cross-correlation is used to de-

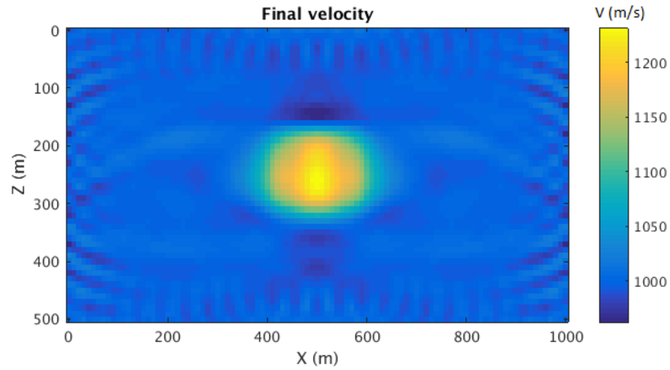

Figure 3: Final velocity model

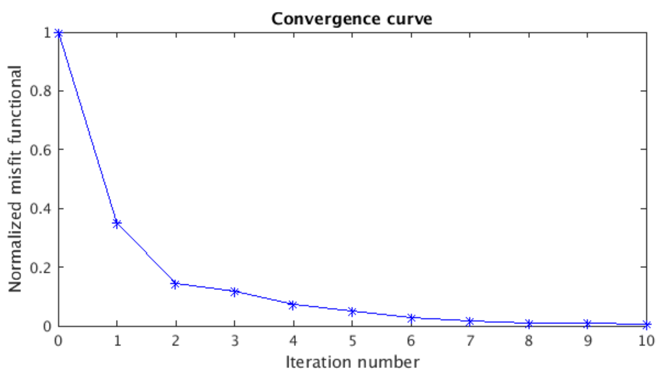

Figure 4: Convergence Curve

termine the traveltime difference, which is influenced by low-frequency waves. Second, the Frechét derivative in wave equation traveltime inversion is a fat ray, which was calculated by reverse-time-migration. Traditional traveltime tomograph uses a ray-tracing method to find the ray, assuming infinite frequency content.

In reality, the best 'match' of predicted data and observed data doesn't necessarily give the maximum cross-correlation. It is better to mute all but first arrivals, and then use the cross-correlation to find traveltime differences.

We never get such good coverage in field data, and that actually leaves strong foot print of acquisition.

REFERENCES

Luo, Y., and G. T. Schuster, 1991, Wave-equation traveltime inversion: Geophysics, **56**, 645–653.

Maarten, V., and R. D. van Der Hilst, 2005, On sensitivity kernels for wave-equationtransmission tomography: Geophysical Journal International, **160**, 621–633.

Tarantola, A., 1984, Inversion of seismic reflection data in the acoustic approximation: Geophysics, **49**, 1259–1266.